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AN ITERATIVE METHOD FOR APPROXIMATING AVERAGE COST OPTIMAL (s,S) INVENTORY POLICIES

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An Iterative Method for Approximating Average Cost Optimal (s,S) Inventory Policies

by

H.C. Tijms

Summary. This paper considers the dynamic inventory model with a discrete demand. There is a constant lead time, backlogging of excess demand, a fixed set-up cost, and holding and shortage costs whose negatives are unimodal. The criterion is the long-run average cost. An iterative method with a varying appropriately chosen discount factor is studied. This iterative method supplies policies of the (s,S) type and convergent upper and lower bounds on the minimal average cost. Further, the average cost of the (s_n,S_n) policy found at the n-th iteration lies between the corresponding upper and lower bound. Also, for all n sufficiently large the (s_n,S_n) policy is average cost optimal. Computational considerations are given for the special case of linear holding and shortage costs.

1. The inventory model and preliminaries

We consider an inventory model in which the demands for a single item in periods 1,2,... are independent, identically distributed discrete random variables. Let $\phi(j)$ be the probability of demand j in a period, $(j=0,1,\ldots)$. It is assumed that the demand distribution $\{\phi(j)\}$ has a finite, positive mean μ. Any unfilled demand in a period is completely backlogged. At the beginning of each period the stock on hand and on order is reviewed. At each review an order may be placed for any positive integral amount of stock. An order placed at the beginning of period t is delivered at the beginning of period $t+\lambda$, where λ is a fixed non-negative integer. The demand in each period takes place after review and delivery (if any). The stock on hand and on order may take on any integral value, where a negative value indicates the existence of a backlog.

The following costs are considered. The cost of ordering j units is $K\delta(j)$, where $K \geq 0$, $\delta(0) = 0$, and $\delta(j) = 1$ for $j \geq 1$. *) Let g(j) be the (expected) holding and shortage costs in a period when j is the stock on hand at the beginning of that period just after any additions to stock. It is assumed that g(j) is non-negative. Future costs are not discounted. For any integer k, let

$$L(k) = \sum_{j=0}^{\infty} g(k-j)\, \phi^{\lambda}(j),$$

*) A linear purchase cost c contributes an amount $c\mu$ to the average cost of of any policy to be considered, so for the average cost criterion we may take $c = 0$.

where $\phi^0(0) = 1$, $\phi^0(j) = 0$ for $j \geq 1$ and, for $n \geq 1$, $\phi^n(j)$ is the n-fold convolution of $\phi(j)$ with itself, i.e., $\phi^n(j)$ is the probability of a cumulative demand j in n periods. We assume that L(k) is finite for all k. Clearly, L(k) represents the expected holding and shortage costs in period $t+\lambda$ when k is the stock on hand and on order at the beginning of period t just after ordering. The following conditions are imposed on L(k): (a) the function - L(k) is unimodal, i.e., there is an integer S_0 such that $L(k) \leq L(k-1)$ for $k \leq S_0$ and $L(k+1) \geq L(k)$ for $k \geq S_0$; (b) $L(k) > K+L(S_0)$ for all $|k|$ sufficiently large. Define r as the smallest integer such that

$$L(r) \leq K + L(S_0),$$

and let R be the largest integer for which

$$L(R) \leq K + L(S_0).$$

For the infinite period model an (s,S) policy is a stationary policy of the following form: If, at review, the stock on hand and on order $i < s$, order S-i units; otherwise, do not order. Denote by a(s,S) the long-run average expected cost per period when an (s,S) policy is used. It is known that the quantity a(s,S) is independent of the initial stock and is given by [*Iglehart*, 1963; *Tijms*, 1972; *Veinott-Wagner*, 1965]

$$a(s,S) = \{L(S) + \sum_{k=0}^{S-s} L(S-k)\, m(k) + K\}/\{1 + M(S-s)\},$$

where m(k) is defined by $m(k) = \phi(k) + \sum_{j=0}^{k} m(k-j)\, \phi(j)$ and $M(k) = \sum_{j=0}^{k} m(j)$,

$k \geq 0$. Let

$$g = \min\{a(s,S) \mid s,S \text{ integers}, s \leq S\}.$$

A policy for controlling the stock is called *average cost optimal* when it minimizes the long run average expected cost per period for each initial stock. It is known that among the class of all possible policies there is an average cost optimal policy which is of the (s,S) type [*Johnson*, 1968; *Tijms*, 1972] (this result was first proved by *Iglehart* [1963] for the case where L(k) is convex, see also [*Veinott-Wagner*, 1965, pp. 530-531]. Hence the minimal average expected cost is independent of the initial stock and equals g. Moreover, there is an average cost optimal (s,S) policy with $r \leq s \leq S \leq R$ [*Tijms*, 1972; *Veinott-Wagner*, 1965]. In [*Johnson*, 1968] and [*Veinott-Wagner*, 1965] computational methods for finding an average cost optimal (s,S) policy are given. These methods are direct ones and bear on the minimization of the quantity a(s,S). In this paper we shall discuss another approach which is based on the method of successive approximations.

Let $\{\alpha_n, n=1,2,\ldots\}$ be a sequence with $0 \leq \alpha_n \leq 1$ for all n. Define $f_0(i) = 0$ for each integer i, and, for $n = 1,2,\ldots$, define for any integer i,

$$f_n(i) = \inf_{k \geq i}\{K\delta(k-i) + L(k) + \alpha_n \sum_{j=0}^{\infty} f_{n-1}(k-j)\,\phi(j)\}. \tag{1}$$

Then [*Veinott*, 1966], for $n = 1,2,\ldots$,

$$f_n(i) = \begin{cases} K + L(S_n) + \alpha_n \sum_{j=0}^{\infty} f_{n-1}(S_n - j)\ \phi(j) & \text{for } i < s_n, \\ \\ L(i) + \alpha_n \sum_{j=0}^{\infty} f_{n-1}(i-j)\ \phi(j) & \text{for } i \geq s_n, \end{cases} \qquad (2)$$

where S_n is the smallest integer for which $G_n(k) = L(k) + \alpha_n \sum_{j=0}^{\infty} f_{n-1}(k-j)\phi(j)$ is minimal and s_n is the smallest integer such that $G_n(s_n) \leq K + G_n(S_n)$. The result in (2) was first proved by *Scarf* [1960] for the case where $L(k)$ is convex (observe that for our model there is no difference between *Veinott's* formulation of the salvage cost in the finite period model and *Scarf's* one, since the linear purchase cost is zero). Further, we have [*Veinott*, 1966],

$$r \leq s_n \leq S_n \leq R \qquad \text{for } n = 1,2,\ldots\ . \qquad (3)$$

2. Approximations for average cost optimal (s,S) policies

In this section we shall prove that the recursion in (2) supplies upper and lower bounds L_n and U_n such that $L_n \leq g \leq a(s_n,S_n) \leq U_n$ for all n. Moreover, under certain conditions, both L_n and U_n converge as $n \to \infty$ to the minimal average cost g, and for all n sufficiently large the (s_n,S_n) policy is average cost optimal.

We introduce the following conditions.

Condition 1:

(i) $0 < \alpha_n < 1$ for $n = 1,2,\ldots$; (ii) $\alpha_n \to 1$ as $n \to \infty$;

(iii) $\alpha_1 \alpha_2 \ldots \alpha_n \to 0$ as $n \to \infty$; and (iv) $\sum_{j=2}^{n} (\alpha_n \alpha_{n-1} \ldots \alpha_j)|\alpha_j - \alpha_{j-1}| \to 0$ as $n \to \infty$.

Condition 2:

(i) $\alpha_n = 1$ for $n = 1,2,\ldots$, and (ii) $\phi(i) > 0$ for some $i > R-r$.

Remark 1:

It is readily verified that Condition 1 is satisfied when $\alpha_n = 1 - (n+1)^{-b}$ for all n provided that $1/2 < b \leq 1$ (see also [*Hordijk-Tijms*, 1973a]).

Given the sequence $\{\alpha_n\}$, we define

$$\gamma_0 = 0 \quad \text{and} \quad \gamma_n = 1 + \alpha_n \gamma_{n-1} \quad \text{for } n = 1,2,\ldots \tag{4}$$

Observe that $\gamma_n = n$ for all n when $\alpha_n = 1$ for all n. We shall need the following theorem.

Theorem 1:

If Condition 1 or Condition 2 is satisfied, then

$$\lim_{n\to\infty}\{f_n(i) - \gamma_n g\} \text{ exists and is finite for all } i = r-1,\ldots,R. \tag{5}$$

Proof:

Let us define a Markovian decision problem which is closely related to the inventory model under consideration. Consider a dynamic system which at times $t = 1,2,\ldots$ is observed to be in one of $R-r+1$ states labeled by $i = r-1,\ldots,R$. After observing state i, an action k is chosen from the set $A(i)$ of possible actions, where $A(i) = \{i,i+1,\ldots,R\}$. If at time t the system is in state i and action k is chosen, then an expected cost $c_i^k = K\delta(k-i) + L(k)$ is incurred and at time $t+1$ the system will be in state j with probability p_{ij}^k, where $p_{ij}^k = \phi(k-j)$ for $j \neq r-1$ and $p_{i,r-1}^k = \sum_{h>k-r} \phi(h)$ with $\phi(m) = 0$ for $m < 0$.

By (2) and (3) we have $f_n(i) = f_n(r-1)$ for all $i < r$ and $n \geq 0$, so, by (1) and (2),

$$f_n(i) = \min_{k\in A(i)} \{c_i^k + \alpha_n \sum_{j=r-1}^{R} f_{n-1}(j)\, p_{ij}^k\} \quad \text{for } r-1 \leq i \leq R \text{ and } n \geq 1.$$

Further, using the fact that for the inventory model there is an average cost optimal policy of the (s,S) type with $r \leq s \leq S \leq R$, it is easily seen that for the above Markovian decision problem the minimal average expected cost per unit time is independent of the initial state and equals g. It now follows from Theorem 1 in [*Hordijk-Tijms*, 1973a] that under Condition 1 the result (5) holds. Assume now that Condition 2 is satisfied. Then, for any stationary policy f adding to each state i an action $f(i) \in A(i)$, the associated Markov chain $((p_{ij}^{f(i)}))$ has a single recurrent class and has no periodic states. It now follows from [*Lanery*, 1967, Theorem 3 on p. 43] that $f_n(i) - ng$ has a finite limit as $n \to \infty$ for all $r-1 \leq i \leq R$ (see also [*Odoni*, 1969]). This ends the proof.

The proof of the next theorem is an adaptation of proofs given in [*Hordijk-Tijms*, 1973a, 1973b].

Theorem 2:

For any $n \geq 1$, let

$$L_n = \min\{f_n(i) - \alpha_n f_{n-1}(i) \mid r_n - 1 \leq i \leq R\},$$

$$U_n = \max\{f_n(i) - \alpha_n f_{n-1}(i) \mid r_n - 1 \leq i \leq S_n\},$$

where $r_1 = s_1$ and $r_n = \min(s_{n-1}, s_n)$. Then

(a) $L_n \leq g \leq a(s_n, S_n) \leq U_n$ for all $n \geq 1$.

If Condition 1 or Condition 2 is satisfied, then

(b) Both L_n and U_n converge as $n \to \infty$ to g.

(c) For all n sufficiently large the (s_n, S_n) policy is average cost optimal.

Proof:

We first introduce some notation. Let $F = \{(s,S) \mid r \leq s \leq S \leq R\}$. For any (s,S) policy from F, define for $i,j = r-1,\ldots,S$

$$p_{ij}(s,S) = \phi(i-j),\ (i \geq s, j \neq r-1), \quad p_{i,r-1}(s,S) = \sum_{k > i-r} \phi(k),\ (i \geq s),$$

$$p_{ij}(s,S) = \phi(S-j),\ (i < s, j \neq r-1), \text{ and } p_{i,r-1}(s,S) = \sum_{k > S-r} \phi(k),\ (i < s).$$

Denote by $\{\pi_i(s,S),\ i = r-1,\ldots,S\}$ the unique stationary probability distribution of the Markov matrix $((p_{ij}(s,S)))$. Let $c_i(s,S) = K + L(S)$ for $i < s$, and let $c_i(s,S) = L(i)$ for $i \geq s$. Clearly, for all $j = r-1,\ldots,S$,

$$\pi_j(s,S) = \sum_{i=r-1}^{S} \pi_i(s,S)p_{ij}(s,S), \text{ and } a(s,S) = \sum_{i=r-1}^{S} c_i(s,S)\pi_i(s,S). \tag{6}$$

For part (a), fix $n \geq 1$. By (1) and (2) we have for any $(s,S) \in F$,

$$f_n(i) \leq \begin{cases} K + L(S) + \alpha_n \sum_{j=0}^{\infty} f_{n-1}(S-j)\ \phi(j) & \text{for } r-1 \leq i < s, \\ \\ L(i) + \alpha_n \sum_{j=0}^{\infty} f_{n-1}(i-j)\ \phi(j) & \text{for } s \leq i \leq S, \end{cases} \tag{7}$$

with equality for all i when $s = s_n$ and $S = S_n$. Since $f_{n-1}(i) = f_{n-1}(r-1)$ for $i < r$, we can write (7) in the equivalent form

$$f_n(i) \leq c_i(s,S) + \alpha_n \sum_{j=r-1}^{S} f_{n-1}(j)\ p_{ij}(s,S) \quad \text{for } r-1 \leq i \leq S, \tag{8}$$

with equality for all i when $s = s_n$ and $S = S_n$. By (2) we have $f_n(i) - \alpha_n f_{n-1}(i)$ is constant for $i < r_n$, so $f_n(i) \geq \alpha_n f_{n-1}(i) + L_n$ for all $i \leq R$. Choose now an (s,S) policy from F. Then, by (8),

$$\alpha_n f_{n-1}(i) + L_n \leq c_i(s,S) + \alpha_n \sum_{j=r-1}^{S} f_{n-1}(j)\ p_{ij}(s,S) \text{ for } r-1 \leq i \leq S. \tag{9}$$

Now multiply both sides of (9) by $\pi_i(s,S)$ and sum over i. Using (6), we find

$$\alpha_n \sum_i f_{n-1}(i)\ \pi_i(s,S) + L_n \leq a(s,S) + \alpha_n \sum_j f_{n-1}(j)\ \pi_j(s,S),$$

so $L_n \leq a(s,S)$. This implies $L_n \leq g$, since $g = a(s,S)$ for some $(s,S) \in F$. Consider now the (s_n,S_n) policy. When $s = s_n$ and $S = S_n$ the equality sign holds in (8) for all i. Further, $f_n(i) \leq \alpha_n f_{n-1}(i) + U_n$ for all $i \leq S_n$. Hence

$$\alpha_n f_{n-1}(i) + U_n \geq c_i(s_n,S_n) + \alpha_n \sum_{j=r-1}^{S_n} f_{n-1}(j)\ p_{ij}(s_n,S_n) \text{ for } r-1 \leq i \leq S_n,$$

from which we derive in the same way as above that $U_n \geq a(s_n,S_n)$.

(b) This assertion is an immediate consequence of (4) and (5).

(c) By (3) there is an integer N with the following property: if $s = s_m$ and $S = S_m$ for some $m \geq N$, then $s = s_n$ and $S = S_n$ for infinitely many values of n. Let (s^*,S^*) be such that $s_n = s^*$ and $S_n = S^*$ for some $n \geq N$. Choose an infinite sequence $\{n_k\}$ such that $s_{n_k} = s^*$ and $S_{n_k} = S^*$ for all k. When $s = s_n$ and $S = S_n$ the equality sign holds in (8) for all i. Using (4), it now follows that for all k and all $i = r-1,\dots,S^*$

$$f_{n_k}(i) - \gamma_{n_k} g = c_i(s^*,S^*) - g + \alpha_{n_k} \sum_{j=r-1}^{S^*} \{f_{n_k-1}(j) - \gamma_{n_k-1} g\}\ p_{ij}(s^*,S^*).$$

Letting $k \to \infty$, we find $v(i) = c_i(s^*,S^*) - g + \sum_j v(j)\ p_{ij}(s^*,S^*)$ for $r-1 \leq i \leq S$, where v(i) denotes the limit in (5). Multiplying both sides of this equality by $\pi_i(s^*,S^*)$, summing over i, and using (6), we find $a(s^*,S^*) = g$, so the (s^*,S^*) policy is average cost optimal.

We note that the proofs of the Theorems 1 and 2 exploit the fact that there is no linear purchase cost. For the case where a linear purchase cost is included the assertions of Theorem 2 were proved in [*Hordijk-Tijms*, 1973b] under the assumptions: (a) $\alpha_n = 1$ for all n; (b) $\phi(i) > 0$ for all i sufficiently large.

3. Computational considerations

In this section we give some general findings for the convergence of the algorithm (2) for the cases $\alpha_n = 1$ and $\alpha_n = 1 - (n+1)^{-b}$ where $1/2 < b \leq 1$. We assume that the holding and shortage costs in a period are given by $h.\max(i,0) + p.\min(i,0)$ when i is the stock on hand at the end of that period, where $h > 0$ and $p > 0$. Then [cf. *Veinott-Wagner*, 1965],

$$L(k) = p\{(\lambda+1)\mu - k\} + (h+p) \sum_{j<k} \Phi^{\lambda+1}(j),$$

where $\Phi^{\lambda+1}(j) = 0$ for $j < 0$ and $\Phi^{\lambda+1}(j) = \phi^{\lambda+1}(0) + \ldots + \phi^{\lambda+1}(j)$ for $j \geq 0$. It will be clear that the knowledge of the distribution function $\Phi^{\lambda+1}(.)$ is sufficient to compute the bounds r and R. The functions L(k) and $f_n(k)$ need be computed only for $r-1 \leq k \leq R$ (see (2) and (3)). We have investigated two types of demand distribution $\phi(.)$: An arbitrary distribution with $\phi(j) > 0$ for finitely many values of j, and a Poisson distribution.

In the computer program the algorithm was stopped when $(U_n-L_n)/L_n \leq \varepsilon$, which implies $a(s_N,S_N) \leq (1+\varepsilon)g$ where N is defined as the first value of n for which the convergence criterion $(U_n-L_n)/L_n \leq \varepsilon$. The number ε was chosen equal to 0.05, 0.01, or 0.005.

Considerable computer experimentation has demonstrated that for the algorithm with $\alpha_n = 1 - (n+1)^{-b}$ the convergence of $(U_n - L_n)/L_n$ becomes substantially worser as b decreases, where $1/2 < b \leq 1$. In most examples tested where the algorithm with $\alpha_n = 1$ achieved convergence the number of iterations required for this algorithm was considerably less than the number of iterations required for the algorithm with $\alpha_n = 1 - (n+1)^{-1}$. For the algorithm with $\alpha_n = 1$ the criterion $(U_n - L_n)/L_n$ exhibited a strong tendency to decrease exponentially with the number of iterations when the demand is Poisson distributed. This agrees with theoretical results in [*White*, 1963]. For the algorithm with $\alpha_n = 1 - (n+1)^{-b}$ our numerical results indicate that $(U_n - L_n)/L_n = O(1/n^{2b-1})$, as might be expected from theoretical results in [*Hordijk-Tijms*, 1973a]. Finally, there did not seem to be any clear relation between the number of iterations required for convergence and the value of $R - r + 1$; in fact the convergence depends on the probability $\sum_{k>R-r} \phi(k)$.

In table 1 we summarize some numerical results for two examples with an arbitrary demand distribution, where the other parameters are given by (cf. [*Wagner*, p. A32]),

$$K = 24,\ h = 4,\ p = 10, \text{ and } \lambda = 0.$$

Except the results in the columns with entry N', the results in table 1 refer to the algorithm with $\alpha_n = 1 - (n+1)^{-1}$. The integer N' denotes the number of iterations required for the algorithm with $\alpha_n = 1 - (n+1)^{-0.6}$. For each case from table 1 the latter algorithm produced the same policy at the last iteration as the algorithm with $\alpha_n = 1 - (n+1)^{-1}$ did at the final iteration. For the example with $\phi(3) = 1$ the algorithm with $\alpha_n = 1$ leads to $L_n = 12$, $U_n = 24$ for $n \geq 4$, $(s_n,S_n) = (1,3)$ for n odd and $(s_n,S_n) = (2,6)$ for n even. We have $a(1,3) = 24$ and $a(2,6) = 18$; the policy (2,6) is average cost optimal. For the example with $\phi(4) = \phi(5) = 0.5$ the algorithm with $\alpha_n = 1$ leads to $L_n = 19.5$, $U_n = 26$ for $n \geq 3$, $(s_n,S_n) = (2,5)$ for n odd and $(s_n,S_n) = (3,9)$ for n even. We have $a(2,5) = 26$ and $a(3,9) = 22.75$; the policy (3,9) is average cost optimal

Table 1. $\alpha_n = 1 - (n+1)^{-1}$ and $\alpha_n = 1 - (n+1)^{-0.6}$

$\phi(3)=1$; r=1, R=11				$\phi(4)=\phi(5)=0.5$; r=2, R=11			
ε	N	N'	(s_N,S_N)	ε	N	N'	(s_N,S_N)
0.05	27	249	(2,6)	0.05	20	169	(3,9)
0.01	133	>500	(2,6)	0.01	106	>500	(3,9)
0.005	267		(2,6)	0.005	211		(3,9)

In the tables 2 and 3 we summarize some numerical results for a number of examples with a demand distribution of the Poisson type, where the other parameters are given by (cf. [*Wagner–O'Hagan–Lundh*, 1965, p. 697]),

$$K = 64,\ h = 1,\ p = 9,\ \text{and}\ \lambda = 0. \tag{10}$$

The cases I, II and III correspond to $\varepsilon = 0.05$, $\varepsilon = 0.01$ and $\varepsilon = 0.005$. The results in the tables 2 and 3 refer to the algorithm with $\alpha_n = 1$, except the results in the rows with entry N'. The integer N' denotes the number of iterations required for the algorithm with $\alpha_n = 1 - (n+1)^{-1}$. When $\varepsilon = 0.05$ the latter algorithm produced the policies (2,23) and (6,36) for $\mu = 4$ and $\mu = 9$ at the last iteration; for each other case we found the same policy as we did for the algorithm with $\alpha_n = 1$. Each policy from the tables 2 and 3 is average cost optimal, except the policy (12,51) for $\mu = 16$.

Table 2. $\alpha_n = 1$ and $\alpha_n = 1 - (n+1)^{-1}$

	μ	1	2	4	9	16
	r,R	-6,67	-5,68	-3,71	2,78	9,87
Case I	N	75	39	24	20	20
	N'	118	83	59	39	29
	(s_N, S_N)	(0,11)	(1,16)	(2,24)	(6,37)	(12,51)
Case II	N	82	43	36	32	32
	N'	>500	410	290	193	146
	(s_N, S_N)	(0,11)	(1,16)	(2,24)	(6,37)	(12,52)
Case III	N	85	45	42	37	38
	N'	>500	>500	>500	386	291
	(s_N, S_N)	(0,11)	(1,16)	(2,24)	(6,37)	(12,52)

Table 3. $\alpha_n = 1$ and $\alpha_n = 1 - (n+1)^{-1}$.

	μ	20	25	36	49	64
	r,R	13,92	17,98	28,110	41,125	56,142
Case I	N	17	29	113	75	3
	N'	26	27	21	18	17
	(s_N, S_N)	(15,62)	(20,56)	(30,79)	(42,106)	(56,74)
Case II	N	34	69	200	170	3
	N'	130	118	104	91	82
	(s_N, S_N)	(15,62)	(20,56)	(30,79)	(42,106)	(56,74)
Case III	N	41	86	237	211	3
	N'	261	236	207	182	164
	(s_N, S_N)	(15,62)	(20,56)	(30,79)	(42,106)	(56,74)

It is interesting to note that we found that the number of iterations needed to achieve convergence is very small when $\sum_{k>R-r} \phi(k)$ is close enough to 1 (see also Theorem 3 in [*Veinott-Wagner*, 1965]).

Finally, we note that it seems difficult to choose a stop criterion based on reiteration of policies. Several examples were encountered in which s_n and S_n did not change for a number of successive values of n whereas no convergence in policy was achieved. In table 4 we give an example of this phenomenon for the algorithm with $\alpha_n = 1 - (n+1)^{-1}$, where the demand is Poisson distributed with mean $\mu = 4$ and the other parameters are given in (10).

Table 4. $\mu = 4$, $\alpha_n = 1 - (n+1)^{-1}$

n	s_n, S_n	$a(s_n, S_n)$
8- 9	2,20	22.483
10- 11	2,21	22.325
12- 23	2,22	22.224
24-168	2,23	22.173
$\geq$169	2,24	22.166

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